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Conformal invariance and corrections to finite-size scaling: applications to the three-state Potts model

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Abstract. Corrections to finite-size scaling are determined numerically for several levels of the three-state Potts quantum chain with various boundary conditions. It is found that the leading correction term behaves like $N^{-0.8}$. In the case of periodic and twisted boundary conditions the coefficients of the $N^{-0.8}$ term are determined by the three-point correlation functions of the conformal theory.

1. Introduction

In two previous letters (von Gehlen and Rittenberg 1986a, b) we have given the operator content of the finite-size scaling limit of the spectrum of the three-state Potts quantum chain at the critical point. Various boundary conditions have been considered and in each case we have identified the irreducible representations (IR) of the Virasoro algebra which build up the spectra. In this paper we will consider in detail the problem of the finite N correction terms (Privman and Fisher 1983, 1984, Luck 1985, Cardy 1986a, b, Henkel 1987, Reinicke 1986). We first summarise our previous results.

We consider the Hamiltonian of the Potts quantum chain

$$H = -\frac{2}{3\sqrt{3}} \sum_{i=1}^{N} [\sigma_i + \sigma_i^+ + \lambda(\Gamma_i \Gamma_{i+1}^+ + \Gamma_i^+ \Gamma_{i+1})] \tag{1.1}$$

where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \omega = e^{2\pi i/3}. \tag{1.2}$$

Here λ has the meaning of the inverse of the temperature and N represents the number of sites. The Hamiltonian is self-dual and has a critical point at $\lambda = 1$. The normalisation factor $2/3\sqrt{3}$ which fixes the Euclidean timescale is taken from von Gehlen *et al* (1986). We specify the boundary conditions. If we take

$$\Gamma_{N+1} = \omega^{\tilde{Q}} \Gamma_1 \tag{1.3}$$

we denote the corresponding Hamiltonians by $H^{(\tilde{Q})}$. For free boundary conditions we have

$$\Gamma_{N+1} = 0 \tag{1.4}$$

and the corresponding Hamiltonian is $H^{(F)}$. Since the Hamiltonian (1.1) is Z_3 symmetric, each of the matrices $H^{(\tilde{Q})}$ and $H^{(F)}$ has a block-diagonal form: $H^{(\tilde{Q})}$ contains the matrices $H_Q^{(\tilde{Q})}$ and $H^{(F)}$ the matrices $H_Q^{(F)}$ ($Q = 0, 1$ and 2). At $\lambda = 1$ self-duality and the invariance under charge conjugation of the Hamiltonian (1.1) give the following relations among the spectra of the matrices $H_Q^{(\tilde{Q})}$ and $H_Q^{(F)}$:

$$H_Q^{(\tilde{Q})} = H_{\tilde{Q}}^{(Q)} \qquad H_1^{(\tilde{Q})} = H_2^{(\tilde{Q})} \qquad H_1^{(F)} = H_2^{(F)} \tag{1.5}$$

and we are thus left with five independent spectra: $H_0^{(0)}$, $H_1^{(0)}$, $H_1^{(1)}$, $H_0^{(F)}$ and $H_1^{(F)}$.

In the case of periodic ($H_0^{(0)}$ and $H_1^{(0)}$) and twisted ($H_1^{(1)}$) boundary conditions we can further prediagonalise the matrices using the translational invariance of the Hamiltonian. We denote by

$$E_Q^{(\tilde{Q})}(P; s) \qquad P = 0, 1, 2 \ldots; s = 0, 1, 2 \ldots \tag{1.6}$$

the eigenvalues of the matrices $H_Q^{(\tilde{Q})}$ corresponding to the momentum P. Here $s = 0$ corresponds to the lowest eigenvalue, $s = 1$ to the next higher one, etc. $E_0^{(0)}(0; 0) \equiv E_0^{(P)}$ corresponds to the ground-state energy of the Hamiltonian with periodic boundary conditions. The eigenvalues of $H_Q^{(F)}$ are denoted by $E_Q^{(F)}(s)$, $s = 0, 1, 2, \ldots$. $E_0^{(F)}(0) \equiv E_0^{(F)}$ denotes the ground-state energy of the Hamiltonian with free boundary conditions.

We now consider the following quantities which are relevant for finite-size scaling (Cardy 1986a):

$$\mathscr{E}_Q^{(\tilde{Q})}(P; s) = \lim_{N\to\infty} \frac{N}{2\pi}(E_Q^{\tilde{Q}}(P; s) - E_0^{(P)})$$

$$\mathscr{E}_Q^{(F)}(s) = \lim_{N\to\infty} \frac{N}{\pi}(E_Q^{(F)}(s) - E_0^{(F)}). \tag{1.7}$$

It is a consequence of conformal invariance that the spectra given by equations (1.7) can be described by IR of the Virasoro algebra with a central charge depending on the universality class of the system. For the three-state Potts model the central charge is $c = \frac{4}{5}$ (Friedan *et al* 1984, Dotsenko 1984). We denote by Δ the highest weight, and by $\Delta + r$ the level of r having a degeneracy $d(\Delta, r)$ of one IR of the Virasoro algebra. The spectra $\mathscr{E}_Q^{(\tilde{Q})}(P; s)$ are given by the products of two IR (Δ and $\bar{\Delta}$) of two commuting Virasoro algebras with the same central charge. The contribution of one IR $(\Delta, \bar{\Delta})$ to the spectrum $\mathscr{E}_Q^{(\tilde{Q})}(P; s)$ is

$$\mathscr{E}_Q^{(\tilde{Q})}(\Delta + r, \bar{\Delta} + \bar{r}) = \Delta + r + \bar{\Delta} + \bar{r}$$

$$P = (\Delta + r) - (\bar{\Delta} + \bar{r}) - \tfrac{1}{3}\tilde{Q}Q \qquad r, \bar{r} = 0, 1, 2, \ldots . \tag{1.8}$$

The degeneracy of the level $\mathscr{E}_Q^{(\tilde{Q})}(\Delta + r, \bar{\Delta} + \bar{r})$ is $d(\Delta, r)d(\bar{\Delta}, \bar{r})$.

The spectra $\mathscr{E}_Q^{(F)}(s)$ are described by the IR of only one Virasoro algebra, namely an IR (Δ) gives a contribution

$$\mathscr{E}_Q^{(F)} = \Delta + r \qquad r = 0, 1, 2, \ldots \tag{1.9}$$

with a degeneracy $d(\Delta, r)$. (The degeneracies $d(\Delta, r)$ can be computed using the character formulae of Rocha-Caridi (1985).)

It was established by numerical studies (von Gehlen and Rittenberg 1986a, b) and confirmed by analytical calculations (Cardy 1986b, c, Itzykson and Zuber 1986, Zuber 1986) that the spectra $\mathscr{E}_Q^{(\tilde{Q})}$ and $\mathscr{E}_Q^{(F)}$ can be described by the following sums of IR of the Virasoro algebra:

$$\begin{aligned}
&\mathscr{E}_0^{(0)}(0, 0) \oplus (\tfrac{2}{5}, \tfrac{2}{5}) \oplus (\tfrac{7}{5}, \tfrac{2}{5}) \oplus (\tfrac{2}{5}, \tfrac{7}{5}) \oplus (\tfrac{7}{5}, \tfrac{7}{5}) \oplus (3, 0) \oplus (0, 3) \oplus (3, 3) \\
&\mathscr{E}_1^{(0)} = (\tfrac{1}{15}, \tfrac{1}{15}) \oplus (\tfrac{2}{3}, \tfrac{2}{3}) \\
&\mathscr{E}_1^{(1)} = (\tfrac{2}{5}, \tfrac{1}{15}) \oplus (0, \tfrac{2}{3}) \oplus (\tfrac{7}{5}, \tfrac{1}{15}) \oplus (3, \tfrac{2}{3}) \\
&\mathscr{E}_1^{(2)} = (\tfrac{1}{15}, \tfrac{2}{5}) \oplus (\tfrac{2}{3}, 0) \oplus (\tfrac{1}{15}, \tfrac{7}{5}) \oplus (\tfrac{2}{3}, 3)
\end{aligned} \tag{1.10a}$$

and

$$\mathscr{E}_0^{(F)} = (0) \oplus (3)$$
$$\mathscr{E}_1^{(F)} = (\tfrac{2}{3}). \tag{1.10b}$$

With this background in mind we can now formulate the problem we want to clarify in this paper. Through the identification provided by equations (1.8) and (1.9), we can label the levels for finite chains instead of $E_Q^{(\tilde{Q})}(P; s)$ by $E(\Delta + r, \bar{\Delta} + \bar{r}; i)$ in the case of periodic and twisted boundary conditions and analogously for free boundary conditions by $E(\Delta + r; i)$ instead of $E_Q^{(F)}(s)$. Here $i = 1, 2, \ldots, d(\Delta, r)d(\bar{\Delta}, \bar{r})$ for the periodic and twisted boundary conditions and $i = 1, 2, \ldots, d(\Delta, r)$ for the free boundary conditions case. By convention, if the level is not degenerate, we will drop the index i. Now we consider the quantities

$$\mathscr{F}(\Delta + r, \bar{\Delta} + \bar{r}; i) = (N/2\pi)(E(\Delta + r, \bar{\Delta} + \bar{r}; i) - E_0^{(P)})$$
$$\mathscr{F}(\Delta + r; i) = (N/\pi)(E(\Delta + r; i) - E_0^{(F)}) \tag{1.11}$$

which depend on N as do the $E(\Delta + r, \bar{\Delta} + \bar{r}; i)$, $E_0^{(P)}$, etc.

Obviously, the $\mathscr{E}_Q^{(\tilde{Q})}(P; s)$, $\mathscr{E}_Q^{(F)}(s)$ of equation (1.7) are the limits $N \to \infty$ of the corresponding $\mathscr{F}$. We shall study how for large but finite N the $\mathscr{F}(\Delta + r, \bar{\Delta} + \bar{r}; i)$ approach $\Delta + r + \bar{\Delta} + \bar{r}$. Assuming power corrections we write

$$\mathscr{F}(\Delta + r, \bar{\Delta} + \bar{r}; i) = \Delta + r + \bar{\Delta} + \bar{r} + c_1 N^{-\alpha_1} + c_2 N^{-\alpha_2} + \ldots \tag{1.12}$$

where $\alpha_2 > \alpha_1$. The exponents α_1 and α_2 and the coefficients c_1 and c_2 will in general depend on the irreducible representation $(\Delta, \bar{\Delta})$, of the level $(r, \bar{r})$ and of the index i. It was Cardy (1986a, b) who pointed out that conformal invariance can give us much information on the α_j and c_j $(j = 1, 2, \ldots)$. We can do a similar study for the $\mathscr{F}$ corresponding to the free boundary conditions case:

$$\mathscr{F}(\Delta + r, i) = \Delta + r + c_1 N^{-\alpha_1} + c_2 N^{-\alpha_2} + \ldots. \tag{1.13}$$

There is, for the time being, no theory which can explain the finite-size corrections in the case of free boundary conditions.

The paper is organised as follows. In § 2 we present the raw numerical data for the $\mathscr{F}$ corresponding to several levels. We then show the results of the fits which give the corresponding c and α appearing in equations (1.12) and (1.13). In § 3 we show, following Cardy (1986a, b), that the α are related to the scaling dimensions of some conformal operators and that the c are related to expansion coefficients (Belavin *et al* 1984).

In § 4 we compare the numerical results of § 2 with the theoretical predictions of § 3. Our results are summarised in § 5. In the appendix we derive some expansion coefficients from the four-point functions of Dotsenko (1984). These expansion coefficients are used in the calculation of the corrections to finite-size scaling.

2. Numerical studies of the corrections to finite-size scaling

In order to familiarise the reader with the numerical problems involved in our study, in tables 1 and 2 we give the values of $\mathscr{F}(\Delta + r, \bar{\Delta} + \bar{r}; i)$ and $\mathscr{F}(\Delta + r; i)$ defined in equation (1.11) for several levels. Chains from 2 up to 14 sites have been considered for periodic and twisted boundary conditions and from 2 up to 12 sites for free boundary

Table 1. $\mathscr{F}(\Delta + r, \bar{\Delta} + \bar{r}; i)$ defined in equation (1.11) for several levels and for various number of sites N. At the bottom of the table ($N = \infty$) we give the large-N estimates computed using the Van den Broeck–Schwartz approximants.

N	$(\frac{2}{5}, \frac{2}{5})$	$(\frac{1}{15}, \frac{1}{15})$	$(\frac{1}{15}+1, \frac{1}{15})$	$(\frac{2}{3}, \frac{2}{3})$	$(\frac{2}{5}, \frac{1}{15})$	$(0, \frac{2}{3})$
3	0.860 929 187 429	0.139 727 774 637	0.975 320 442 667	1.188 957 756 944	0.440 624 670 027	0.627 829 136 707
4	0.857 345 162 666	0.137 536 701 432	1.045 259 891 347	1.241 533 826 641	0.447 855 523 550	0.645 034 392 164
5	0.852 084 712 808	0.136 443 073 086	1.078 394 256 822	1.269 405 512 273	0.451 890 295 377	0.652 994 085 932
6	0.847 242 474 641	0.135 799 194 200	1.096 459 205 633	1.285 997 222 709	0.454 453 220 281	0.657 291 435 201
7	0.843 101 626 873	0.135 378 678 038	1.107 299 681 174	1.296 707 568 193	0.456 223 755 942	0.659 862 519 852
8	0.839 604 394 604	0.135 083 793 256	1.114 270 659 338	1.304 045 395 972	0.457 520 625 330	0.661 517 648 822
9	0.836 639 908 683	0.134 866 036 304	1.118 992 853 373	1.309 306 601 823	0.458 512 304 935	0.662 643 192 193
10	0.834 105 907 381	0.134 698 811 379	1.122 323 920 292	1.313 216 535 919	0.459 295 957 677	0.663 441 916 745
11	0.831 919 344 044	0.134 566 398 210	1.124 750 941 341	1.316 207 605 865	0.459 931 465 535	0.664 028 359 579
12	0.830 015 070 329	0.134 458 943 442	1.126 566 565 954	1.318 551 000 763	0.460 457 714 981	0.664 471 104 213
13	0.828 342 308 451	0.134 369 973 044	1.127 954 889 855	1.320 424 044 556	0.460 901 047 691	0.664 813 210 325
14	0.826 861 322 116	0.134 295 065 509		1.321 946 801 444		
∞	0.820 (3)	0.1333 (1)	1.1344 (5)	1.3333 (5)	0.4667 (3)	0.666 66 (3)

Table 2. $\mathcal{F}(\Delta + r; i)$ defined in equation (1.11) for several levels and for various number of sites N. At the bottom of the table ($N = \infty$) we give the large-N estimates computed using the Van den Broeck–Schwartz approximants.

N	$(0+2)$	$(0+3)$ or (3)	$(\frac{2}{3})$	$(\frac{2}{3}+1)$	$(\frac{2}{3}+2; 1)$	$(\frac{2}{3}+3; 1)$	$(\frac{2}{3}+3; 2)$
2	1.407 619 279 38	1.806 467 430 53	0.471 151 359 20	1.071 342 236 64			
3	1.595 746 496 25	2.124 844 131 80	0.518 809 930 89	1.250 968 933 29		1.950 003 590 45	2.430 995 247 79
4	1.693 620 757 61	2.307 918 527 14	0.546 486 826 56	1.347 121 267 18	2.106 221 740 46	2.433 997 536 21	2.736 732 293 84
5	1.752 800 948 98	2.425 778 902 92	0.564 719 794 79	1.406 303 210 78	2.205 179 974 43	2.728 445 113 06	2.925 230 986 08
6	1.792 247 186 56	2.507 806 259 82	0.577 715 354 12	1.446 248 683 96	2.273 264 843 38	2.918 601 756 85	3.050 922 862 50
7	1.820 371 694 86	2.568 172 644 74	0.587 489 876 21	1.475 001 873 33	2.322 940 273 20	3.049 046 119 73	3.140 177 127 90
8	1.841 430 971 57	2.614 484 622 28	0.595 134 605 58	1.496 695 943 34	2.360 794 920 65	3.142 932 716 27	3.206 499 847 96
9	1.857 795 730 47	2.651 170 723 63	0.601 293 541 84	1.513 659 628 38	2.390 620 075 89	3.213 154 994 27	3.257 595 454 45
10	1.870 886 230 74	2.680 977 809 75	0.606 372 272 23	1.527 300 868 64	2.414 745 492 59	3.267 340 775 11	3.298 111 004 44
11	1.881 603 371 87	2.705 697 918 84	0.610 639 503 94	1.538 519 506 66	2.434 679 496 58	3.310 239 109 52	3.330 999 227 38
12	1.890 545 485 59	2.726 549 241 70	0.614 280 634 43	1.547 916 994 65	2.451 440 859 77	3.344 936 576 67	3.358 218 624 33
∞	2.000 (5)	2.98 (3)	0.6662 (4)	1.668 (2)	2.66 (1)	3.64 (4)	3.66 (2)

conditions. The large N limit for the $\mathcal{F}$ computed using Van den Broeck-Schwartz (1979) approximants is shown at the bottom of the tables. Comparing the values of the estimates with the expected values (see equations (1.12) and (1.13)) one gets a feeling of the errors involved. We would like to make a remark on the application of the Van den Broeck-Schwartz approximants to our problem. The approximants are defined as follows. Assume we have a set of numbers $A_1, A_2, \ldots, A_N$ converging to a limit A. We denote by $[n, L]$, $L=0, 1, \ldots$, new sets defined through the equations

$$\begin{aligned}
&[n, -1] = \infty \\
&[n, 0] = A_n \\
&([n, L+1]-[n, L])^{-1} \\
&\qquad = -\beta([n, L-1]-[n, L])^{-1} \\
&\qquad\quad +([n+1, L]-[n, L])^{-1}+([n-1, L]-[n, L])^{-1} \qquad (2.1)
\end{aligned}$$

where β is a parameter. It was pointed out by Hamer and Barber (1981) that if the corrections terms to A are power behaved, the approximants are stable for $\beta=-1$. This stability for $\beta=-1$ was observed not only for the approximants shown in tables 1 and 2, but also for the other calculations which are described in this section.

We now consider the correction terms to the $\mathcal{F}$ (see equations (1.12) and (1.13)). We start with the $\mathcal{F}(\Delta+r, \bar{\Delta}+\bar{r}; i)$. We first determine α_1^{eff}. In order to do so, we consider the estimates

$$(\alpha_1)_N = \{\ln[(N+1)/N]\}^{-1} \ln\left(\frac{\mathcal{F}(\Delta+r, \bar{\Delta}+\bar{r}; i)_N-(\Delta+r+\bar{\Delta}+\bar{r})}{\mathcal{F}(\Delta+r, \bar{\Delta}+\bar{r}; i)_{N+1}-(\Delta+r+\bar{\Delta}+\bar{r})}\right) \qquad (2.2)$$

where N represents the number of sites. From the estimates $(\alpha_1)_N$ one obtains α_1^{eff} using Van den Broeck-Schwartz approximants. The results are shown in table 3. In order to exemplify how the approximants work, in table 4 we give the approximants

Table 3. Estimates for c_1, c_2, α_1 and α_2 defined by equation (1.12) (periodic and twisted boundary conditions) for several levels. The values of α_1 and α_2 assumed in the determination of c_1 and c_2 are also specified.

$(\Delta+r, \bar{\Delta}+\bar{r})$	α_1^{eff}	c_1	α_2^{eff}	c_2
$(0+2, 0)$	2.01 (2)	−5.810 (1) ($\alpha_1=2.0$)	—	—
$(\frac{1}{15}, \frac{1}{15})$	0.796 (10)	0.006 57 (2) ($\alpha_1=0.8$)	1.98 (5)	0.032 38 (2) ($\alpha_2=2.1$)
$(\frac{1}{15}+1, \frac{1}{15})$	1.71 (5)	0.034 (2) ($\alpha_1=0.8$)		−1.6 (1) ($\alpha_2=2$)
$(\frac{2}{5}, \frac{2}{5})$	0.7998 (3)	0.2364 (2) ($\alpha_1=0.8$)	1.80 (2)	−0.328 (2) ($\alpha_2=2$)
$(\frac{2}{5}, \frac{1}{15})$	0.82 (3)	−0.0395 (3) ($\alpha_1=0.8$)	1.80 (6)	−0.15 (3) ($\alpha_2=2$)
$(\frac{2}{5}, \frac{1}{15}+1)$	1.79	>-0.2 ($\alpha_1=0.8$) −3.417 (4) ($\alpha_1=1.8$)		
$(\frac{2}{3}, \frac{2}{3})$	1.691 (2)	−0.708 (1) ($\alpha_1=1.6$)		

Table 4. Van den Broeck-Schwartz approximants with $\beta=-1$ for α_1^{eff} corresponding to the level $(\frac{2}{5},\frac{2}{5})$.

	L					
n	0	1	2	3	4	5
1	0.210 732 0594					
2	0.431 188 8305	0.628 096 9357				
3	0.535 197 8824	0.676 365 8152	0.796 517 6465			
4	0.595 084 1840	0.703 913 3105	0.798 030 1924	0.799 491 6997		
5	0.633 713 5729	0.721 661 2838	0.798 767 5181	0.799 733 5156	0.799 991 6545	
6	0.660 553 8650	0.734 011 8063	0.799 183 3625	0.799 843 7650	0.800 048 0630	0.800 024 2323
7	0.680 211 5652	0.743 080 7920	0.799 437 5114	0.799 908 2095	0.799 996 0275	
8	0.695 186 8521	0.750 010 1406	0.799 602 0556	0.799 942 5950		
9	0.706 949 1907	0.755 469 4091	0.799 712 7415			
10	0.716 416 4606	0.759 876 5940				
11	0.724 190 2939					

for α_1^{eff} in the case of $\mathcal{F}(\frac{2}{5},\frac{2}{5})$. We notice from table 3 that the values of α_1^{eff} cluster into two groups. For the levels $(\frac{1}{15},\frac{1}{15})$, $(\frac{2}{5},\frac{2}{5})$, $(\frac{2}{5},\frac{1}{15})$ one finds $\alpha_1^{\text{eff}}\approx 0.8$ and for the others one finds $\alpha_1^{\text{eff}}\approx 1.8$-$2.0$. Since large values of α_1^{eff} can also be obtained from $\alpha_1=0.8$ and $\alpha_2=2$ with c_1 and c_2 having opposite signs (see equation (1.12)), we have assumed $\alpha_1=0.8$ for all the levels and have determined c_1. It turns out that for the level $(\frac{1}{15}+1,\frac{1}{15})$ we find very stable approximants and less stable approximants for $(\frac{2}{5},\frac{1}{15}+1)$. The ansatz $\alpha_1=0.8$ does not work for the levels $(0+2,0)$ and $(\frac{2}{3},\frac{2}{3})$. The values of the c_1 are shown in table 3. Next we determine α_2^{eff}. We did it only for the levels where $\alpha_1\approx 0.8$. The corresponding values of α_2^{eff} are also shown in table 3. They are around 1.8-2.0. Next we have assumed $\alpha_2=2.0$ and have determined the c_2. In § 4 we will discuss the interpretation of the results shown in table 3.

We now turn to the problem of the corrections to finite-size scaling in the case of free boundary conditions. We have determined α_1^{eff} first (see equation (1.13)). The estimates for α_1^{eff} computed using the equivalent of equation (2.2) are shown in table 5. At the bottom of the table are the Van den Broeck-Schwartz approximants for α_1^{eff}. To our surprise they all cluster around $\alpha_1=0.8$. We have done a fit to the $\mathcal{F}$ taking $\alpha_1=0.8$ and assuming $\alpha_2=2$:

$$\mathcal{F}(\Delta+r;\, i)=\Delta+r+c_1N^{-0.8}+c_2N^{-2}. \tag{2.3}$$

The values for c_1 and c_2 are shown in table 6. There is not yet a theory which can explain the value for α_1 or the values of the c_1 in the case of free boundary conditions but we shall return to this problem in another publication.

3. Breaking of conformal invariance

In this section we essentially follow Cardy (1986a, b). We consider first some relations which can be obtained using conformal invariance for the two- and three-point correlation functions.

In a conformal invariant theory in two dimensions, each primary field $\varphi(X, Y)$ is related to the highest weights $(\Delta, \bar{\Delta})$ of the tensor product of two irreducible representations of two commuting Virasoro algebras. The two-point correlation function is

Table 5. Values of $(\alpha_1)_N$ defined by equation (2.2) for several levels (free boundary conditions). At the bottom of the table ($N = \infty$) we give the large-N estimates computed using the Van den Broeck-Schwartz approximants.

N	$(0+2)$	$(0+3)$ or (3)	$(\frac{2}{3})$	$(\frac{2}{3}+1)$	$(\frac{2}{3}+2; 1)$	$(\frac{2}{3}+3; 1)$	$(\frac{2}{3}+3; 2)$
2	0.942 392 73	0.765 221 77	0.689 072 54	0.885 685 40			
3	0.963 627 92	0.815 825 33	0.720 429 02	0.914 408 86		1.151 272 61	0.988 980 98
4	0.961 846 31	0.836 632 63	0.737 361 55	0.917 888 04	0.870 640 27	1.223 208 12	1.014 491 57
5	0.953 507 16	0.845 440 83	0.747 923 87	0.913 511 82	0.875 495 36	1.242 290 66	1.019 658 86
6	0.943 617 19	0.848 822 41	0.755 144 02	0.906 760 11	0.875 672 87	1.243 040 15	1.015 885 97
7	0.933 860 82	0.849 573 89	0.760 401 34	0.899 577 02	0.873 802 69	1.234 844 01	1.008 320 11
8	0.924 797 59	0.849 005 62	0.764 409 89	0.892 677 65	0.871 061 14	1.222 271 26	0.999 293 48
9	0.916 572 90	0.847 774 52	0.767 575 03	0.886 307 41	0.868 004 69	1.207 692 96	0.989 910 71
10	0.909 175 75	0.846 225 82	0.770 143 51	0.880 523 21	0.864 904 13	1.192 389 32	0.980 701 59
11	0.902 539 15	0.844 546 61	0.772 274 03	0.875 306 34	0.861 891 05	1.177 065 03	0.971 913 10
∞	0.84 (3)	0.82 (2)	0.800 (4)	0.83 (2)	0.81 (4)	0.9 (1)	0.8 (1)

Table 6. Estimates for c_1 and c_2 defined by equation (2.3) (free boundary conditions) for several levels.

Level $(\Delta+r;\, i)$	c_1	c_2
(2)	−0.73 (2)	−1.3 (1)
(3)	−1.943 (5)	−1.2 (1)
$(\frac{2}{3})$	−0.393 (4)	−0.13 (4)
$(\frac{2}{3}+1)$	−0.81 (2)	−0.9 (1)
$(\frac{2}{3}+2;\, 1)$	−1.49 (4)	−1.5 (1)
$(\frac{2}{3}+3;\, 1)$	−1.7 (3)	−13 (3)
$(\frac{2}{3}+3;\, 2)$	−2.0 (2)	−6 (3)

completely determined:

$$\langle \varphi_{\Delta,\bar\Delta}(z_1,\bar z_1)\varphi_{\Delta,\bar\Delta}(z_2,\bar z_2)\rangle = (z_1-z_2)^{-2\Delta}(\bar z_1-\bar z_2)^{-2\bar\Delta} \tag{3.1}$$

where

$$z = X + \mathrm{i}Y \qquad \bar z = X - \mathrm{i}Y.$$

The quantities $x=\Delta+\bar\Delta$ and $s=\Delta-\bar\Delta$ are called the scaling dimensions and the spin of the field $\varphi_{\Delta,\bar\Delta}$. Note that the right-hand side of equation (3.1) fixes the normalisation of the field $\varphi_{\Delta,\bar\Delta}$. The three-point function of primary fields is also fixed by conformal invariance:

$$\begin{aligned}\langle \varphi_{\Delta_1,\bar\Delta_1}(z_1,\bar z_1)&\varphi_{\Delta_2,\bar\Delta_2}(z_2,\bar z_2)\varphi_{\Delta_3,\bar\Delta_3}(z_3,\bar z_3)\rangle\\ &= c_{\Delta_1,\Delta_2,\Delta_3} c_{\bar\Delta_1,\bar\Delta_2,\bar\Delta_3}(z_1-z_2)^{\Delta_3-\Delta_1-\Delta_2}(\bar z_1-\bar z_2)^{\bar\Delta_3-\bar\Delta_1-\bar\Delta_2}(z_2-z_3)^{\Delta_1-\Delta_2-\Delta_3}\\ &\quad\times(\bar z_2-\bar z_3)^{\bar\Delta_1-\bar\Delta_2-\bar\Delta_3}(z_3-z_1)^{\Delta_2-\Delta_3-\Delta_1}(\bar z_3-\bar z_1)^{\bar\Delta_2-\bar\Delta_3-\bar\Delta_1}.\end{aligned} \tag{3.2}$$

The $c_{\Delta_1,\Delta_2,\Delta_3}$ are called expansion coefficients and they are also fixed by the conformal theory (Belavin *et al* 1984).

Under a conformal transformation $w=w(z)$ the correlation function of primary fields transforms as follows ($w_1=w(z_1)$, etc):

$$\langle \varphi_{\Delta_1,\bar\Delta_1}(w_1,\bar w_1)\ldots\rangle = (w'(z_1))^{-\Delta_1}(\overline{w'(z_1)})^{-\bar\Delta_1}\langle\varphi_{\Delta_1,\bar\Delta_1}(z_1,\bar z_1)\ldots\rangle. \tag{3.3}$$

We now consider the conformal transformation

$$w = (N/2\pi)\ln z = \tau + \mathrm{i}v \tag{3.4}$$

which maps the plane into the strip ($-\frac{1}{2}N\leqslant v\leqslant\frac{1}{2}N$, $-\infty<\tau<\infty$). As the result of the transformation (3.4), the two-point function has the following expression on the strip:

$$\begin{aligned}\langle\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)&\varphi_{\Delta,\bar\Delta}(v_2,\tau_2)\rangle\\ &=(2\pi/N)^{2x}\xi^{\Delta}\bar\xi^{\bar\Delta}(1-\xi)^{-2\Delta}(1-\bar\xi)^{-2\bar\Delta}\\ &=(2\pi/N)^{2x}\sum_{r,\bar r=0}^{\infty}a_r(2\Delta)a_{\bar r}(2\bar\Delta)\\ &\quad\times\exp[-(2\pi/N)(x+r+\bar r)\tau-(2\pi\mathrm{i}/N)(s+r-\bar r)v]\end{aligned} \tag{3.5}$$

where

$$\xi = z_1/z_2 \qquad \tau=\tau_2-\tau_1 \qquad v = v_2-v_1 \tag{3.6}$$

and

$$a_r(\alpha)=\frac{\Gamma(\alpha+r)}{r!\,\Gamma(\alpha)}. \tag{3.7}$$

For the three-point functions of two primary fields $\varphi_{\Delta,\bar\Delta}$ and one spin-zero primary field $\varphi_{\Delta_1,\Delta_1}$ (this is the quantity of interest for our calculations) we have

$$\begin{aligned}\langle\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)\varphi_{\Delta_1,\Delta_1}(v_2,\tau_2)\varphi_{\Delta,\bar\Delta}(v_3,\tau_3)\rangle\\ = c_{\Delta,\Delta,\Delta_1}c_{\bar\Delta,\bar\Delta,\Delta_1}(2\pi/N)^{2x+x_1}(\xi_1\xi_2)^{2\Delta}(\bar\xi_1\bar\xi_2)^{2\bar\Delta}(|1-\xi_1|\,|1-\xi_2|)^{-2\Delta_1}\\ \times(1-\xi_1\xi_2)^{\Delta_1-2\Delta}(1-\bar\xi_1\bar\xi_2)^{\Delta_1-2\bar\Delta}\end{aligned} \tag{3.8}$$

where

$$\xi_1=\exp(2\pi/N)(w_1-w_2)\qquad \xi_2=\exp(2\pi/N)(w_2-w_3). \tag{3.9}$$

We now write the two-point function (3.5) using the spectral decomposition and equations (1.7) and (1.8):

$$\begin{aligned}\langle\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)\varphi_{\Delta,\bar\Delta}(v_2,\tau_2)\rangle&=\langle 0|\hat\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)\hat\varphi_{\Delta,\bar\Delta}(v_2,\tau_2)|0\rangle\\ &=\langle 0|\exp(-H\tau_1-\mathrm{i}\hat P v_1)\hat\varphi_{\Delta,\bar\Delta}(0,0)\exp(-H\tau-\mathrm{i}\hat P v)\hat\varphi_{\Delta,\bar\Delta}(0,0)\\ &\quad\times\exp(H\tau_2+\mathrm{i}\hat P v_2)|0\rangle\\ &=\sum_{i,r,\bar r}\langle 0|\hat\varphi_{\Delta,\bar\Delta}(0,0)|\Delta+r,\bar\Delta+\bar r;i\rangle\langle\Delta+r,\bar\Delta+\bar r;i|\hat\varphi_{\Delta,\bar\Delta}(0,0)|0\rangle\\ &\quad\times\exp[-(2\pi/N)(x+r+\bar r)\tau-(2\pi\mathrm{i}/N)(s+r-\bar r)v]\end{aligned} \tag{3.10}$$

where H is the Hamiltonian, $\hat P$ is the momentum operator and the summation over i is over the various degeneracies. We now compare equations (3.5) and (3.10) and obtain

$$\begin{aligned}&\langle 0|\hat\varphi_{\Delta,\bar\Delta}(0,0)|\Delta,\bar\Delta\rangle\langle\Delta,\bar\Delta|\hat\varphi_{\Delta,\bar\Delta}(0,0)|0\rangle=(2\pi/N)^{2x}\\ &\langle 0|\hat\varphi_{\Delta,\bar\Delta}(0,0)|\Delta+1,\bar\Delta\rangle\langle\Delta+1,\bar\Delta|\hat\varphi_{\Delta,\bar\Delta}(0,0)|0\rangle=2\Delta(2\pi/N)^{2x}.\end{aligned} \tag{3.11}$$

The three-point function in this limit can also be written using the spectral decomposition. One finds

$$\begin{aligned}\langle\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)\varphi_{\Delta_1,\Delta_1}(v_2,\tau_2)\varphi_{\Delta,\bar\Delta}(v_3,\tau_3)\rangle\\ = c_{\Delta,\Delta,\Delta_1}c_{\bar\Delta,\bar\Delta,\Delta_1}(2\pi/N)^{2x+x_1}(\xi_1\xi_2)^{2\Delta}(\bar\xi_1\bar\xi_2)^{2\bar\Delta}\\ \times[1+(\Delta_1^2-\Delta_1+2\Delta)\xi_1\xi_2+\ldots].\end{aligned} \tag{3.12}$$

The three-point function in this limit can also be written using the spectral decomposition. One finds

$$\begin{aligned}\langle 0|\hat\varphi_{\Delta,\bar\Delta}(v_1,\tau_1)\hat\varphi_{\Delta_1,\Delta_1}(v_2,\tau_2)\hat\varphi_{\Delta,\bar\Delta}(v_3,\tau_3)|0\rangle\\ =(\xi_1\xi_2)^{\Delta}(\bar\xi_1\bar\xi_2)^{\bar\Delta}(\langle 0|\hat\varphi_{\Delta,\bar\Delta}(0,0)|\Delta,\bar\Delta\rangle\langle\Delta,\bar\Delta|\hat\varphi_{\Delta,\bar\Delta}(0,0)|0\rangle\\ \times\langle\Delta,\bar\Delta|\hat\varphi_{\Delta_1,\Delta_1}(0,0)|\Delta,\bar\Delta\rangle+\xi_1\xi_2\langle 0|\hat\varphi_{\Delta,\bar\Delta}(0,0)|\Delta+1,\bar\Delta\rangle\\ \times\langle\Delta+1,\bar\Delta|\hat\varphi_{\Delta,\bar\Delta}(0,0)|0\rangle\langle\Delta+1,\bar\Delta|\hat\varphi_{\Delta_1,\Delta_2}(0,0)|\Delta+1,\bar\Delta\rangle+\ldots).\end{aligned} \tag{3.13}$$

We compare equations (3.12) and (3.13) and with the help of equations (3.11) obtain

$$\langle\Delta,\bar\Delta|\hat\varphi_{\Delta_1,\Delta_1}(0,0)|\Delta,\bar\Delta\rangle=(2\pi/N)^{x_1}c_{\Delta,\Delta,\Delta_1}c_{\bar\Delta,\bar\Delta,\Delta_1} \tag{3.14a}$$

$$\langle\Delta+1,\bar\Delta|\hat\varphi_{\Delta_1,\Delta_1}(0,0)|\Delta+1,\bar\Delta\rangle=(2\pi/N)^{x_1}[1+(\Delta_1^2-\Delta_1)/2\Delta]c_{\Delta,\Delta,\Delta_1}c_{\bar\Delta,\bar\Delta,\Delta_1}. \tag{3.14b}$$

We now assume that the conformal invariant theory described by the Hamiltonian H is perturbed by an additional term:

$$\tilde{H} = H + g\int_{-N/2}^{N/2} \hat{\varphi}_{\Delta_1,\Delta_1}(v, 0)\,\mathrm{d}v \tag{3.15}$$

where $\tilde{H}$ is the new Hamiltonian and g is a coupling constant. We apply standard perturbation theory and stop at the first order:

$$E^{(P)} = \langle 0|\tilde{H}|0\rangle = \langle 0|H|0\rangle = E_{\mathrm{c}}^{(P)} \tag{3.16a}$$

$$\begin{aligned} E(\Delta+r, \bar{\Delta}+\bar{r}; i) &= \langle\Delta+r, \bar{\Delta}+\bar{r}; i|\tilde{H}|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle \\ &= E_{\mathrm{c}}(\Delta+r, \bar{\Delta}+\bar{r}; i) + Ng\langle\Delta+r, \bar{\Delta}+\bar{r}; i|\hat{\varphi}_{\Delta_1,\Delta_1}(0,0)|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle. \end{aligned} \tag{3.16b}$$

Here $E_{\mathrm{c}}^{(P)}$ and $E_{\mathrm{c}}(\Delta, \bar{\Delta}; i)$ are the eigenvalues of the unperturbed Hamiltonian. From equations (3.16*a, b*) and (1.11), we obtain

$$\begin{aligned} \mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}; i) &= \Delta+r+\bar{\Delta}+\bar{r} \\ &\quad + (N^2/2\pi)g\langle\Delta+r, \bar{\Delta}+\bar{r}; i|\hat{\varphi}_{\Delta_1,\Delta_1}(0,0)|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle. \end{aligned} \tag{3.17}$$

If we specialise to the levels considered in equations (3.14*a, b*), we obtain

$$\mathscr{F}(\Delta, \bar{\Delta}) = \Delta+\bar{\Delta}+[g(2\pi)^{x_1-1}N^{2-x_1}]c_{\Delta,\Delta,\Delta_1}c_{\bar{\Delta},\bar{\Delta},\Delta_1} \tag{3.18a}$$

$$\mathscr{F}(\Delta+1, \bar{\Delta}) = \Delta+1+\bar{\Delta}+[g(2\pi)^{x_1-1}N^{2-x_1}][1+(\Delta_1^2-\Delta_1)/2\Delta]c_{\Delta,\Delta,\Delta_1}c_{\bar{\Delta},\bar{\Delta},\Delta_1}. \tag{3.18b}$$

From equations (3.18*a, b*) we learn that in the first order in $[gN^{2-x_1}(2\pi)^{x_1-1}]$ the $\mathscr{F}$ can be obtained from the knowledge of the three-point function of the conformal theory. Reinicke (1986) has shown that the higher-order corrections can be obtained from the n-point correlation functions of the conformal theory (the four-point function determines the quadratic correction, etc).

In equation (3.15) we made the hypothesis that the perturbation is given by the primary operator $\hat{\varphi}_{\Delta_1,\Delta_1}(v, \tau)$. It is interesting to consider instead the operator $\hat{\varphi}_{2,2}(v, \tau)$ which corresponds to the descendents $r = \bar{r} = 2$ of the unit operator ($\Delta_1 = \bar{\Delta}_1 = 0$). In this case one obtains instead of (3.18*a*)

$$\mathscr{F}(\Delta, \bar{\Delta}) = \Delta+\bar{\Delta}+g(2\pi)^3N^{-2}[(\Delta-\tfrac{1}{24}c\delta_{\Delta,0})(\bar{\Delta}-\tfrac{1}{24}c\delta_{\bar{\Delta},0})-(\tfrac{1}{24}c)^2]. \tag{3.19}$$

This result was obtained by Reinicke (1986). Notice that $\hat{\varphi}_{2,2}(v, \tau)$ gives 'analytic' contributions to the $\mathscr{F}$. With the equations (3.18) and (3.19) at hand we can now try to give an interpretation to the results obtained in § 2.

4. Comparison of the predictions of conformal invariance and the numerical fits

We have seen in § 2 that the correction terms to the $\mathscr{F}$ given by equation (1.12) can be described by fits of the form

$$\mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}) = \Delta+\bar{\Delta}+r+\bar{r}+A_1N^{-0.8}+A_2N^{-2}+\dots \tag{4.1}$$

where the values of the coefficients A_1 and A_2 for various levels can be obtained from table 3. We first consider the $N^{-0.8}$ correction term in equation (4.1). From equations (3.18) we learn that

$$\Delta_1 = \tfrac{1}{2}x_1 = \tfrac{7}{5} \tag{4.2}$$

and that the leading correction to finite-size scaling is indeed given by the next to leading thermal exponent (Privman and Fisher 1983). In order to derive the coefficients A_1 in equation (4.1), we use equations (3.18) to which one has to add the numerical values of the expansion coefficients $c_{\Delta,\Delta,7/5}$. From Belavin *et al* (1984) we learn which expansion coefficients $c_{\Delta_1,\Delta_2,\Delta_3}$ are different from zero; they are shown in table 7. From this table we obtain

$$c_{0,0,7/5} = c_{2/3,2/3,7/5} = 0 \tag{4.3}$$

and thus the A_1 for the levels $(0+2,0)$ and $(\frac{2}{3},\frac{2}{3})$ have to vanish in agreement with table 3. In order to obtain the remaining non-vanishing coefficients we use the four-point function of Dotsenko (1984) and obtain (see the appendix)

$$(c_{2/5,2/5,7/5})^2 = \frac{6}{7}\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{2}{5})}\right)^{3/2}\left(\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})}\right)^{1/2} \tag{4.4}$$

and

$$(c_{2/5,2/5,7/5})^2 = 36(c_{1/15,1/15,7/5})^2. \tag{4.5a}$$

As suggested by the results of table 3, we take the following solution of equation (4.5a):

$$c_{2/5,2/5,7/5} = -6c_{1/15,1/15,7/5}. \tag{4.5b}$$

We can now use equation (4.5b) together with equations (3.18a, b) and obtain the A_1 for all levels if we specify one of them (the coupling constant g is unknown). Since the errors for the level $(\frac{2}{5},\frac{2}{5})$ are the smallest, we have used the corresponding value of A_1 in order to determine the others. The expected values for the A_1 (A_1^{exp}) are compared in table 8 with the values obtained from table 3. The agreement between

Table 7. Values of Δ_3 for which the expansion coefficients $c_{\Delta_1,\Delta_2,\Delta_3} = c_{\Delta_2,\Delta_1,\Delta_3}$ are non-zero.

	Δ_2					
Δ_1	(0)	(3)	$(\frac{7}{5})$	$(\frac{2}{5})$	$(\frac{2}{3})$	$(\frac{1}{15})$
(0)	(0)	(3)	$(\frac{7}{5})$	$(\frac{2}{5})$	$(\frac{2}{3})$	$(\frac{1}{15})$
(3)	—	(0)	$(\frac{2}{5})$	$(\frac{7}{5})$	$(\frac{2}{3})$	$(\frac{2}{3})\oplus(\frac{1}{15})$
$(\frac{7}{5})$	—	—	$(0)\oplus(\frac{7}{5})$	$(\frac{2}{5})$	$(\frac{1}{15})$	$(\frac{2}{3})\oplus(\frac{1}{15})$
$(\frac{2}{5})$	—	—	—	$(0)\oplus(\frac{7}{5})$	$(\frac{1}{15})$	$(\frac{2}{3})\oplus(\frac{1}{15})$
$(\frac{2}{3})$	—	—	—	—	$(0)\oplus(3)\oplus(\frac{2}{3})$	$(\frac{7}{5})\oplus(\frac{2}{5})\oplus(\frac{1}{15})$
$(\frac{1}{15})$	—	—	—	—	—	$(0)\oplus(3)\oplus(\frac{7}{5})\oplus(\frac{2}{5})\oplus(\frac{2}{3})\oplus(\frac{1}{15})$

Table 8. Comparison between the values of A_1 computed using conformal invariance (A_1^{eff}) and those determined numerically.

Level $(\Delta+r,\bar{\Delta}+\bar{r})$	A_1^{exp}	A_1
$(0+2,0)$	0	0
$(\frac{1}{15},\frac{1}{15})$	0.006 566	0.006 57 (2)
$(\frac{1}{15}+1,\frac{1}{15})$	0.034 14	0.034 (2)
$(\frac{2}{5},\frac{1}{15})$	−0.039 4	−0.039 5 (3)
$(\frac{2}{5},\frac{1}{15}+1)$	−0.204 9	>-0.2
$(\frac{2}{3},\frac{2}{3})$	0	0

the two sets of A_1 is very good. We have thus shown that the leading correction to the $\mathcal{F}$ can be understood using the calculations of § 2. We now proceed with the second correction term (A_2N^{-2}). This term has to be considered with care. It is clear that a correction term $N^{-1.6}$ should be present (this is the second-order correction coming from the same operator which gave us $N^{-0.8}$ in first order). We can assume that numerically this term is negligible in the interval of N we are considering and again try to determine the coefficients A_2 assuming that one of them is known. This check can be done using equation (3.19). This analysis can be performed using the values of A_2 which can be obtained from table 3. The result is negative. In order to illustrate the point, let us consider the ratio of the A_2 corresponding to the levels $(\frac{2}{5}, \frac{2}{5})$ and $(\frac{1}{15}, \frac{1}{15})$. We find

$$\frac{A_2(\frac{2}{5}, \frac{2}{5})}{A_2(\frac{1}{15}, \frac{1}{15})} = \frac{143}{3} \tag{4.6a}$$

from equation (3.19) and

$$\frac{A_2(\frac{2}{5}, \frac{2}{5})}{A_2(\frac{1}{15}, \frac{1}{15})} = -10.1 \tag{4.6b}$$

from table 3. Many explanations for this mismatch are possible. The most obvious of them is that the A_2N^{-2} terms in our fits are effective representations for combinations of the form

$$c_2N^{-1.6} + c_3N^{-2} + \ldots . \tag{4.7}$$

In that case it is hopeless to establish numerically the separate contributions. We would like also to mention that in the case of the Ising model the 'analytic' correction (N^{-2}) is not given by the operator $\varphi_{2,2}(v, \tau)$ used to derive equation (3.19) but by a descendent of the energy density operator (Reinicke 1986). It is thus possible that in the three-state Potts model there are no (N^{-2}) correction terms present at all.

5. Conclusions

We have analysed numerically the corrections to finite-size scaling for various levels and different boundary conditions. Our results are summarised in tables 3 and 6. In the case of periodic and twisted boundary conditions we show that the leading correction term is given by $N^{-0.8}$ terms. We find the coefficient of the $N^{-0.8}$ corrections to be in excellent numerical agreement with the short-distance expansion coefficients which appear in a first-order perturbation treatment of the breaking of conformal invariance. The next to leading correction term is not yet under control.

In the case of free boundary conditions the leading correction term again behaves like $N^{-0.8}$. The theoretical determination of the coefficients in this case will be published elsewhere.

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Appendix. Determination of the expansion coefficients from the four-point correlation functions

We consider the four-point functions for the energy density ($\varphi_{2/5,2/5}$) and spin ($\varphi_{1/15,1/15}$) operators in the plane (Dotsenko 1984):

$$\langle\varphi_{2/5,2/5}(z_1,\bar z_1)\varphi_{2/5,2/5}(z_2,\bar z_2)\varphi_{2/5,2/5}(z_3,\bar z_3)\varphi_{2/5,2/5}(z_4,\bar z_4)\rangle$$
$$= A\left|\frac{z_{12}z_{34}}{z_{13}z_{32}z_{24}z_{14}}\right|^{8/5}|F(-\tfrac{8}{5},-\tfrac{1}{5},-\tfrac{2}{5};\eta)|^2 + B\frac{|z_{13}z_{32}z_{24}z_{14}|^{6/5}}{|z_{12}z_{34}|^4}|F(\tfrac{6}{5},\tfrac{13}{5},\tfrac{12}{5};\eta)|^2 \tag{A1a}$$

and

$$\langle\varphi_{1/15,1/15}(z_1,\bar z_1)\varphi_{1/15,1/15}(z_2,\bar z_2)\varphi_{2/5,2/5}(z_3,\bar z_3)\varphi_{2/5,2/5}(z_4,\bar z_4)\rangle$$
$$= C\frac{|z_{12}|^{4/3}}{|z_{13}z_{32}z_{24}z_{14}|^{4/5}}|F(-\tfrac{4}{5},\tfrac{3}{5},\tfrac{2}{5};\eta)|^2 + D\frac{|z_{13}z_{32}z_{24}z_{14}|^{2/5}}{|z_{12}|^{16/15}|z_{34}|^{12/5}}|F(\tfrac{2}{5},\tfrac{9}{5},\tfrac{8}{5};\eta)|^2 \tag{A1b}$$

where

$$z_{ij}=z_i-z_j \qquad \eta = z_{13}z_{24}/z_{12}z_{34}$$
$$A=\frac{(\Gamma(\frac{1}{5})\Gamma(\frac{4}{5}))^2}{\Gamma(\frac{3}{5})\Gamma(\frac{2}{5})}(\Gamma(\tfrac{2}{5})\Gamma(\tfrac{3}{5})+\Gamma(\tfrac{1}{5})\Gamma(\tfrac{4}{5}))^{-1}$$
$$B=\frac{36}{49}\frac{(\Gamma(\frac{1}{5}))^3(\Gamma(\frac{3}{5}))^2\Gamma(\frac{4}{5})}{(\Gamma(\frac{2}{5}))^4}(\Gamma(\tfrac{2}{5})\Gamma(\tfrac{3}{5})+\Gamma(\tfrac{1}{5})\Gamma(\tfrac{4}{5}))^{-1} \tag{A2}$$
$$C=\tfrac{49}{144}B \qquad D=\tfrac{4}{9}A$$

and the F denote standard hypergeometric functions. The normalisation of A, B, C, D is chosen to obtain the correct normalisation in equation (3.11).

We perform the conformal transformation (3.4) and use equation (3.3) in order to obtain the correlation functions on the strip. Using

$$\xi_j = z_j/z_{j+1}=\exp\{(2\pi/N)[(\tau_j-\tau_{j+1})+\mathrm{i}(v_j-v_{j+1})]\} \tag{A3}$$

we find in the small ξ limit:

$$\langle\varphi_{2/5,2/5}(v_1,\tau_1)\varphi_{2/5,2/5}(v_2,\tau_2)\varphi_{2/5,2/5}(v_3,\tau_3)\varphi_{2/5,2/5}(v_4,\tau_4)\rangle$$
$$=\left(\frac{2\pi}{N}\right)^{16/5}|\xi_1|^{4/5}|\xi_3|^{4/5}\left[1+\tfrac{4}{25}|\xi_2|^4 + \left(\frac{6}{7}\right)^2\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{2}{5})}\right)^3\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})}|\xi_2|^{14/5}+\ldots\right] \tag{A4a}$$

and

$$\langle\varphi_{1/15,1/15}(v_1,\tau_1)\varphi_{1/15,1/15}(v_2,\tau_2)\varphi_{2/5,2/5}(v_3,\tau_3)\varphi_{2/5,2/5}(v_4,\tau_4)\rangle$$
$$=\left(\frac{2\pi}{N}\right)^{28/15}|\xi_1|^{2/15}|\xi_3|^{4/5}\left[1+\frac{1}{(15)^2}|\xi_2|^4 + \left(\frac{1}{7}\right)^2\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{2}{5})}\right)^3\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})}|\xi_2|^{14/5}+\ldots\right]. \tag{A4b}$$

It is easy to show, applying the methods of § 3, that one has in general:

$$\langle\varphi_{\Delta_1,\bar\Delta_1}(v_1,\tau_1)\varphi_{\Delta_1,\bar\Delta_1}(v_2,\tau_2)\varphi_{\Delta_2,\bar\Delta_2}(v_3,\tau_3)\varphi_{\Delta_2,\bar\Delta_2}(v_4,\tau_4)\rangle$$
$$=\left(\frac{2\pi}{N}\right)^{2x_1+2x_2}\xi_1^{\Delta_1}\bar\xi_1^{\bar\Delta_1}\xi_3^{\Delta_2}\bar\xi_3^{\bar\Delta_2}$$
$$\times\sum_k c_{\Delta_1,\Delta_1,\Delta_k}c_{\Delta_2,\Delta_2,\Delta_k}c_{\bar\Delta_1,\bar\Delta_1,\bar\Delta_k}c_{\bar\Delta_2,\bar\Delta_2,\bar\Delta_k}\,\xi_2^{\Delta_k}\bar\xi_2^{\bar\Delta_k}. \tag{A5}$$

From equations (A4*a*, *b*) and (A5) we obtain

$$(c_{2/5,2/5,7/5})^2=\frac{6}{7}\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{2}{5})}\right)^{3/2}\left(\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})}\right)^{1/2}=36(c_{1/15,1/15,7/5})^2. \tag{A6}$$

The expansion coefficients given in equation (A6) are used in § 4. In a similar way we obtain

$$(c_{1/15,1/15,2/5})^2=\tfrac{7}{12}(c_{2/5,2/5,7/5})^2$$
$$(c_{1/15,2/5,2/3})^2=\tfrac{2}{3}. \tag{A7}$$

The expansion coefficients given in equation (A7) might be useful for other applications.

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